UK Maths Trust

Intermediate Mathematical Olympiad

Cayley paper

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Solutions

These are polished solutions and do not illustrate the process of failed ideas and rough work by which candidates may arrive at their own solutions.

It is not intended that these solutions should be thought of as the 'best' possible solutions and the ideas of readers may be equally meritorious.

Enquiries about the Intermediate Mathematical Olympiad should be sent to:

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1. A number N is a two-digit positive integer. It can be written as $\overline{ab}$ to show its digits a and b. How can N written in terms of a and b in standard algebra?
Find all possible N with digits a and b, where $N = \overline{ab}$ and $N = a + b^2$.

SOLUTION

$$N = 10a + b$$

$$N = 10a + b = a + b^2$$

$$9a = b^2 - b = b(b - 1)$$

Because 9 divides the left-hand side, 9 must divide the right-hand side.

Since b and $b - 1$ have no factors in common, 9 must divide either b or $b - 1$.

If 9 divides b, $b = 0$ or $b = 9$.

If $b = 0$, $a = 0$, but then N is not a two-digit number.

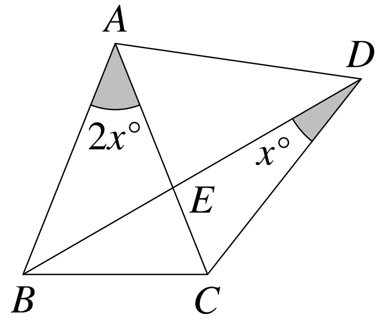
If $b = 9$, $a = 8$, and $N = 89$. This satisfies the conditions of the question.

If 9 divides $b - 1$, $b = 1$.

If $b = 1$, $a = 0$, but then N is not a two-digit number.

The only solution is $N = 89$.

2. In the quadrilateral $ABCD$, $AB = AD$ and $AC = DC$. The diagonals of the quadrilateral meet at E and $AE = EB$.
Also, $\angle BAC = 2\angle BDC$ as shown.
Find the value of $\angle BDC$.
Hence prove that triangle ABC is isosceles.



Solution

$\angle ABE = \angle BAE = \angle BAC = 2x$ because they are base angles of an isosceles triangle.

$\angle ADB = \angle ABD = 2x$ because they are base angles of an isosceles triangle.

$\angle CAD = \angle CDA = \angle CDE + \angle EDA = x + 2x = 3x$ because they are base angles of an isosceles triangle.

In triangle ADB, $2x + 2x + 3x + 2x = 180$ because the sum of the angles in a triangle is 180°.

Therefore, $x = 20$.

This means $\angle CAD = \angle CDA = 60°$.

Therefore, triangle ADC is equilateral.

So, $AC = AD = AB$.

Since $AC = AB$, triangle ABC is isosceles.

3. A cuboid has dimensions 7 by 8 by x where $x > 8$. A water tank is a cylinder with base area, a. The cuboid stays at the bottom when water is added.
When the cuboid has the 8 by x face down, a volume, v, of water is added to the tank to a depth of 7, level with the top of the cuboid.
When the cuboid has the 7 by x face down, the water level drops by 1.
When the cuboid has the 7 by 8 face down, the water level again drops by 1.
Find how much the water level drops when the cuboid is removed.

Solution

The first scenario gives this equation.

$$v = 7(a - 8x)$$

The second scenario gives this equation.

$$v = 6(a - 7x)$$

The third scenario gives this equation.

$$v = 5(a - 56)$$

Equating the first two expressions for v gives this equation.

$$a = 14x$$

Equating the last two equations for v gives this equation.

$$a = 42x - 280$$

Substituting in for a and solving gives $x = 10$.

Substituting in for x gives $a = 140$ and $v = 420$.

With no cuboid in the tank, the height of the water level is $\frac{v}{a} = \frac{420}{140} = 3$.

Therefore, the water drops by a further 2.

4. A tile is one of the two triangles made by cutting a square of area 3 in half along the diagonal. In attempting to tile a 49 by 49 square floor using these tiles with none of them overlapping and none going outside the square, it is impossible to tile it fully. Find the maximum number of these tiles which can be used in this tiling.
Demonstrate how to do this and prove that no more tiles will fit.

Solution

The area of the floor is $49^2 = 2401$.

The area of each triangle is $\frac{3}{2}$.

The maximum number of tiles which can fit is the largest whole number below $2401 \div \frac{3}{2}$, which is 1600.

Placing four tiles with their right angles at the same point gives a square of area 6.

A tiling of 400 of these squares in a 20 by 20 grid is a square of area 2400, which will fit inside the square of side 49, so it is possible to fit 1600 tiles on the floor.

Therefore the maximum number of tiles which can be used is 1600.

5. A keypad is a 3 by 3 grid with the digits 1 to 3 in order on the top row, 4 to 6 in order on the second row and 7 to 9 in order on the third row.
Seb has the keypad upside-down (i.e rotated through 180°) and types his three-digit PIN as if the keypad were the right way up. The number he types in turns out to be k times his original PIN where k is a positive integer. Find all possible PINs and prove there are no others.

Solution

Note that, if the original digit in the PIN is a, the digit typed in the rotated grid is $10 - a$.

Suppose the original PIN is $x = 100a + 10b + c$.

The number typed in is $100(10 - a) + 10(10 - b) + (10 - c)$.

This simplifies to $1110 - 100a - 10b - c = 1110 - x$.

Therefore, $1110 - x = kx$.

So, $1110 = kx + x = (k + 1)x$.

Using the prime factorisation, $(k + 1)x = 2 \times 3 \times 5 \times 37$.

Since x is a three-digit number, $k + 1$ must be at most 11 and be a factor of 1110.

This gives the following options.

$k = 1$ and $x = 555$ with 555 typed into the rotated keypad.

$k = 2$ and $x = 370$, but this cannot be the PIN because there is no zero on the keypad.

$k = 4$ and $x = 222$ with 888 typed into the rotated keypad.

$k = 5$ and $x = 185$ with 925 typed into the rotated keypad.

$k = 9$ and $x = 111$ with 999 typed into the rotated keypad.

The possible PINs are 111, 185, 222 and 555.

6. 2025 penguins (675 each of emperors, kings and gentoos) stand in a line from left to right. They may move by performing the penguin shuffle in which an emperor takes a king's place, a gentoo takes the position vacated by the emperor and the displaced king takes the place vacated by the gentoo.
Prove that, no matter what the starting order, it is possible to place the emperors on the left, kings in the middle and gentoos on the right after a finite number of shuffles.

Solution

Firstly, note that all the gentoos can be placed on the right by the following process.

If there is not a gentoo in a place which should have a gentoo, perform the penguin shuffle once or twice with an emperor and a king until there is a gentoo in that place.

Secondly, note the following set of moves with emperor E and king K.

Choose another emperor e and gentoo g and their current places (e,K,E,g).

Perform a shuffle on e, K and g to get (g,e,E,K).

Perform a shuffle on E, K and g to get (K,e,g,E).

Perform a shuffle on E, K and g to get (E,e,K,g).

There is now a king where there was an emperor and an emperor where there was a king and the other two places have the same type of penguin as before. It does not matter that it is not the same penguin of that type.

Thirdly, note that by performing this set of moves on any king not in the correct place and any emperor not in the correct place, a king and an emperor end in the correct place without affecting whether other penguins are in the correct place.

Finally, note that once all the kings are in the correct place by using this set of moves, the emperors will also be in the correct places and the penguins are in order with all the emperors on the left, kings in the middle and gentoos on the right.

